ПРОЕКТСОФТ ®
34-36 Peyo Yavorov blvd,
Sofia 1164, Bulgaria

tel +359 2 423 4455
proektsoft.bg@gmail.com
https://calcpad.eu

A CALCPAD PROGRAM

FOR ANALYSIS OF PLANE FRAMES WITH VARIABLE CROSS-SECTIONS

(using the finite element method)

by

Eng. Nedelcho Ganchovsky

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I. Introduction

This Calcpad program performs analysis of plane frames with variable cross-sections using the finite element method. The input data is entered in text format as vectors and matrices as follows:

- joint coordinates;
- joint numbers at the ends of the elements;
- material properties;
- dimensions and types of cross-sections;
- support conditions;
- load values.

As a result, diagrams of internal forces and deflections of structural elements are obtained. The schemes are automatically generated by the program, using the SVG graphical format.

II. Calcpad source code

```
#include svg_drawing.cpd
"Analysis of plane frames with variable cross-sections
'<h4>Joint coordinates - xJ; yJ</h4>
#hide
#deg
δz = 10^-12
Precision = 10^-8
x_J = [0; 0; 8; 16; 16]*m
y_J = [0; 8; 10; 8; 0]*m
#show
x_J','y_J
n_J = len(x_J)
'<h4>Elements - [J1; J2]</h4>
#hide
e_J = [1; 2|3; 2|3; 4|5; 4]
#show
transp(e_J)
n_E = n_rows(e_J)
'Element endpoint coordinates
x_1(e) = x_J.e_J.(e; 1)','y_1(e) = y_J.e_J.(e; 1)
x_2(e) = x_J.e_J.(e; 2)','y_2(e) = y_J.e_J.(e; 2)
'Element length - 'l(e) = sqrt((x_2(e) - x_1(e))^2 + (y_2(e) - y_1(e))^2)
'Element directions
c(e) = (x_2(e) - x_1(e))/l(e)','s(e) = (y_2(e) - y_1(e))/l(e)
'Transformation matrix
'Diagonal 3x3 block -'t(e) = [c(e); s(e); 0|-s(e); c(e); 0|0; 0; 1]
'Generation of the full transformation matrix
T(e) = add(t(e); add(t(e); matrix(6; 6); 1; 1); 4; 4)
'<h4>Supports - [Joint; cx; cy; cr]</h4>
#hide
c = [1; 10^20kN/m; 10^20kN/m; 0kNm|5; 10^20kN/m; 10^20kN/m; 10^20kNm]
#show
c
n_c = n_rows(c)
'<h4>Loads - [Element, qx, qy]</h4>
#hide
q = [1; 10kN/m; 0kN/m|2; 0kN/m; -20kN/m|3; 0kN/m; -10kN/m]
n_q = n_rows(q)
q_x = vector(n_E)*kN/m
q_y = vector(n_E)*kN/m
$Repeat{q_x.(q.(i; 1)) = q_x.(q.(i; 1)) + q.(i; 2) @ i = 1 : n_q}
$Repeat{q_y.(q.(i; 1)) = q_y.(q.(i; 1)) + q.(i; 3) @ i = 1 : n_q}
#show
q
'Load values on elements
q_x
q_y
```

```
'<h4>Scheme of the structure</h4>
#hide
w = max(x_J)
h = max(y_J)
W = 240
H = h*W/w
k = W/w
#def svg$ = '<svg viewbox="'-3m*k' '-2m*k' '(w + 6m)*k' '(h + 4m)*k'" xmlns="http://www.w3.org/2000/svg" version="1.1" style="font-family: Georgia Pro; font-size:5pt; width:'W + 150'pt; height:'H + 200*H/W'pt">
#def thin_style$ = style = "stroke:green; stroke-width:1; fill:none"
#def thick_style$ = style = "stroke:green; stroke-width:2; fill:none"
k_q = m/kN
#show
#val
svg$
#for i = 1 : n_E
    #hide
    x1 = x_1(i)*k
    y1 = (h - y_1(i))*k
    x2 = x_2(i)*k
    y2 = (h - y_2(i))*k
    q_xi = q_x.i
    q_yi = q_y.i
    α = atan2(c(i); s(i))
    #if α ≥ 135
        α = α - 180
    #end if
    #if α < -45
        α = α + 180
    #else if α < 0
        α = 360 + α
    #end if
    #if q_xi ≠ 0kN/m
        #hide
        x3 = x2 - q_xi*k_q','y3 = y2
        x4 = x1 - q_xi*k_q','y4 = y1
        x = (x3 + x4)/2 - 5*sign(q_xi)
        y = (y3 + y4)/2
        #show
        '<polygon points="'x1','y1' 'x2','y2' 'x3','y3' 'x4','y4'" style="stroke:magenta; stroke-width:1; stroke-opacity:0.3; fill:magenta; fill-opacity:0.1;" />
        text$(x;y;α;qx='abs(q_xi)')
    #end if
    #if q_yi ≠ 0kN/m
        #hide
        x3 = x2','y3 = y2 + q_yi*k_q
        x4 = x1','y4 = y1 + q_yi*k_q
        x = (x3 + x4)/2
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        y = (y3 + y4)/2 + 5*sign(q_yi)
        #show
        '<polygon points="'x1','y1' 'x2','y2' 'x3','y3' 'x4','y4'" style="stroke:dodgerblue; stroke-width:1; stroke-opacity:0.4; fill:dodgerblue; fill-opacity:0.15;" />
        text$(x;y;α;qy='abs(q_yi)')
    #end if
    #show
    line$(x1; y1; x2; y2; main_style$)
#loop
'<g id="frame">
#for i = 1 : n_E
    #hide
    x1 = x_1(i)*k
    y1 = (h - y_1(i))*k
    x2 = x_2(i)*k
    y2 = (h - y_2(i))*k
    #show
    line$(x1; y1; x2; y2; main_style$)
#loop
#for i = 1 : n_c
    #hide
    j = c.(i; 1)
    x1 = x_J.j*k
    y1 = (h - y_J.j)*k
    δ = w/30*k*sign(x1 - w/2*k)
    x2 = x1 - δ
    y2 = y1 - abs(δ)
    x3 = x1 + δ
    y3 = y1 + abs(δ)
    #show
    #if c.(i; 2) ≠ 0kN/m
        #if c.(i; 3) ≠ 0kN/m
            #if c.(i; 4) ≠ 0kNm
                line$(x1; y1; x1; y3; thin_style$)
                line$(x2; y3; x3; y3; thick_style$)
            #else
                line$(x2; y3; x3; y3; thick_style$)
                line$(x2; y3; x1; y1; thin_style$)
                line$(x3; y3; x1; y1; thin_style$)
            #end if
        #else
            #if c.(i; 4) ≠ 0kNm
                line$(x1; y1; x2; y1; thin_style$)
                line$(x2; y2; x2; y3; thick_style$)
                line$(x2 - δ/2; y2; x2 - δ/2; y3; thick_style$)
            #else
                line$(x2; y2; x1; y1; thin_style$)
                line$(x2; y3; x1; y1; thin_style$)
                line$(x2; y2; x2; y3; thin_style$)
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                line$(x2 - δ/2; y2; x2 - δ/2; y3; thick_style$)
            #end if
        #end if
    #else
        #if c.(i; 3) ≠ 0kN/m
            #if c.(i; 4) ≠ 0kNm
                line$(x1; y1; x1; y3; thin_style$)
                line$(x2; y3; x3; y3; thick_style$)
                line$(x2; y3 + abs(δ)/2; x3; y3 + abs(δ)/2; thick_style$)
            #else
                line$(x2; y3; x3; y3; thin_style$)
                line$(x2; y3; x1; y1; thin_style$)
                line$(x3; y3; x1; y1; thin_style$)
                line$(x2; y3 + abs(δ)/2; x3; y3 + abs(δ)/2; thick_style$)
            #end if
        #else
            line$(x2; y2; x3; y3; thick_style$)
        #end if
    #end if
#loop
'</g>
#for i = 1 : n_E
    #hide
    x = (x_1(i) + x_2(i))*k/2
    y = (h - (y_1(i) + y_2(i))/2)*k
    #show
    texth$(x + 0.8m*sign(W/2 - x)*k; y + 0.6m*k; e'i')
#loop
#for i = 1 : n_J
    point$(x_J.i*k; (h - y_J.i)*k; point_style$)
    texth$((x_J.i - 0.7m*sign(w/2 - x_J.i))*k; (h - y_J.i - 0.4m)*k; J'i')
#loop
dimv$((w + 2m)*k; (h - y_J.4)*k; h*k; 'y_J.4')
dimv$((w + 2m)*k; 0; (h - y_J.4)*k; 'h - y_J.4')
dimh$(0; w*k; (h + 1.5m)*k; 'w')
'</svg>
#equ
'<h4>Materials</h4>
'Modules of elasticity -'E = [45; 35]*GPa
'Poisson coefficients -'v = [0.2; 0.2]
'Shear modules -'G = E/(2*(1 + v))
'Assignments on elements -'e_M = [1; 2; 2; 1]
'<h4>Cross-sections</h4>
#hide
b = vector(2)','h = vector(2)
#show
'Section S1 -'b_1 = 300mm','h_1 = 300mm
'Section S2 -'b_2 = 300mm','h_2 = 900mm
'General representation
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```
'Width -'b(ξ; z) = b_1 + (b_2 - b_1)*ξ
'Height -'h(ξ) = h_1 + (h_2 - h_1)*ξ
'<h4>Cross section properties</h4>
'Area -'A(ξ) = $Area{b(ξ; z) @ z = 0mm : h(ξ)}|cm^2
'First moment of area -'S(ξ) = $Area{b(ξ; z)*z @ z = 0mm : h(ξ)}|cm^3
'Centroid -'z_c(ξ) = S(ξ)/A(ξ)|mm
'Second moment of area -'I(ξ) = $Area{b(ξ; z)*(z - z_c(ξ))^2 @ z = 0mm : h(ξ)}|cm^4
'First moment of area below z -'S_1(ξ; z) = $Area{b(ξ; ζ)*(z_c(ξ) - ζ) @ ζ = 0.1mm : z}
'Shear area -'A_Q(ξ) = I(ξ)^2/$Area{S_1(ξ; z)^2/b(ξ; z) @ z = 0.1mm : h(ξ) - 0.1mm}
'<h4>Element stiffness matrix</h4>
'Elastic properties for element "e"
EA(e; x) = E.e_M.e*A(x/l(e))
EI(e; x) = E.e_M.e*I(x/l(e))
GA_Q(e; x) = G.e_M.e*A_Q(x/l(e))
'Stiffness matrix for element with variable cross-section
'Displacement due to F<sub>x</sub> = 1 in primary system -'u_F(e) = $Area{1/EA(e; x) @ x = 0m : l(e)}
'Displacement due to F<sub>y1</sub> = 1 in primary system, with account of shear deflections -'v_F1(e) = $Area{x^2/EI(e; x) @ x = 0m : l(e)} + $Area{1/GA_Q(e; x) @ x = 0m : l(e)}
'Rotation due to F<sub>y1</sub> = 1 in primary system -'ϕ_F1(e) = -$Area{x/EI(e; x) @ x = 0m : l(e)}' + 0
'Displacement due to M<sub>1</sub> = 1 in primary system -'v_M1(e) = ϕ_F1(e)
'Rotation due to M<sub>1</sub> = 1 in primary system -'ϕ_M1(e) = $Area{1/EI(e; x) @ x = 0m : l(e)}' + 0
'Determinant -'D_1(e) = ϕ_M1(e)*v_F1(e) - ϕ_F1(e)^2
'Displacement due to F<sub>y2</sub> = 1 in primary system -'v_F2(e) = $Area{(l(e) - x)^2/EI(e; x) @ x = 0m : l(e)} + $Area{1/GA_Q(e; x) @ x = 0m : l(e)}
'Rotation due to F<sub>2</sub> = 1 in primary system -'ϕ_F2(e) = $Area{(l(e) - x)/EI(e; x) @ x = 0m : l(e)}' + 0
'Displacement due to M<sub>2</sub> = 1 in primary system -'v_M2(e) = ϕ_F2(e)
'Rotation due to M<sub>2</sub> = 1 in primary system -'ϕ_M2(e) = ϕ_M1(e)
'Determinant -'D_2(e) = ϕ_M2(e)*v_F2(e) - ϕ_F2(e)^2
'3x3 blocks of the stiffness matrix for element "e"
k_ii(e) = [D_1(e)/u_F(e)*kN*m|0; ϕ_M1(e)*kNm; -ϕ_F1(e)*kN|0; -ϕ_F1(e)*kN; v_F1(e)*(kN/m)]*(kN^-2/D_1(e))
k_ij(e) = [-D_2(e)/u_F(e)*kN*m|0; -ϕ_M2(e)*kNm; ϕ_F2(e)*kN|0; (ϕ_F2(e) - ϕ_M2(e)*l(e))*kN; -(v_F2(e) - ϕ_F2(e)*l(e))*(kN/m)]*(kN^-2/D_2(e))
k_ji(e) = [-D_1(e)/u_F(e)*kN*m|0; -ϕ_M1(e)*kNm; ϕ_F1(e)*kN|0; (ϕ_F1(e) + ϕ_M1(e)*l(e))*kN; -(v_F1(e) + ϕ_F1(e)*l(e))*(kN/m)]*(kN^-2/D_1(e))
k_jj(e) = [D_2(e)/u_F(e)*kN*m|0; ϕ_M2(e)*kNm; -ϕ_F2(e)*kN|0; -ϕ_F2(e)*kN; v_F2(e)*(kN/m)]*(kN^-2/D_2(e))
'Full element stiffness matrix
k_E(e) = stack(augment(k_ii(e); k_ij(e)); augment(k_ji(e); k_jj(e)))
```

```
'Stiffness matrices obtained in local coordinates
k_E(1)
k_E(2)
'Stiffness matrices obtained in global coordinates
transp(T(1))*k_E(1)*T(1)
transp(T(2))*k_E(2)*T(2)
'<h4>Global stiffness matrix</h4>
#hide
K = symmetric(3*n_J)
'Add element stiffness matrices
#for e = 1 : n_E
    i = 3*e_J.(e; 1) - 2','j = 3*e_J.(e; 2) - 2
    t = t(e)','tT = transp(t)
    add(tT*k_ii(e)*t; K; i; i)
    #if j > i
        add(tT*k_ij(e)*t; K; i; j)
    #else
        add(tT*k_ji(e)*t; K; j; i)
    #end if
    add(tT*k_jj(e)*t; K; j; j)
#loop
'Add supports
#for i = 1 : n_c
    j = 3*c.(i; 1) - 2
    add(vec2diag(last(row(c; i); 3)/[kN/m; kN/m; kNm]); K; j; j)
#loop
#show
K
'<h4>Element load vector</h4>
'Load functions
'Shear -'q_E(e; x) = -q_x.e*s(e) + q_y.e*c(e)
'Axial -'n_E(e; x) = q_x.e*c(e) + q_y.e*s(e)
'Functions of internal forces in primary system
'Axial forces -'N_0(e; x) = -$Area{n_E(e; ξ) @ ξ = 0m : x}
'Shear forces -'Q_0(e; x) = $Area{q_E(e; ξ) @ ξ = 0m : x}
'Bending moments -'M_0(e; x) = $Area{q_E(e; ξ)*(x - ξ) @ ξ = 0m : x}
'Reactions at element ends
'Displacements along “x“ due to axial loads -'u_n(e) = $Area{N_0(e; x)/EA(e; x) @ x = 0m : l(e)}
'Displacements along “y“ due to lateral loads -'v_q(e) = $Area{M_0(e; x)*x/EI(e; x) @ x = 0m : l(e)} + $Area{Q_0(e; x)/GA_Q(e; x) @ x = 0m : l(e)}
'Rotations due to lateral loads -'ϕ_q(e) = -$Area{M_0(e; x)/EI(e; x) @ x = 0m : l(e)}
'Element endpoint loads in local coordinate system
'For joint “1”:
F_x1(e) = -u_n(e)/u_F(e)
F_y1(e) = (ϕ_M1(e)*v_q(e) - ϕ_F1(e)*ϕ_q(e))/D_1(e)
M_1(e) = (v_F1(e)*ϕ_q(e) - ϕ_F1(e)*v_q(e))/D_1(e)
'For joint “2”:
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```
F_x2(e) = -F_x1(e) - N_0(e; l(e))
F_y2(e) = -F_y1(e) + Q_0(e; l(e))
M_2(e) = -M_1(e) + F_y1(e)*l(e) - M_0(e; l(e))
'Element endpoint loads in global coordinate system
'For joint “1”:
F’_x1(e) = F_x1(e)*c(e) - F_y1(e)*s(e)
F’_y1(e) = F_x1(e)*s(e) + F_y1(e)*c(e)
'For joint “2”:
F’_x2(e) = F_x2(e)*c(e) - F_y2(e)*s(e)
F’_y2(e) = F_x2(e)*s(e) + F_y2(e)*c(e)
'Element load vector
F_E(e) = [F’_x1(e); F’_y1(e); M_1(e)*m^-1; F’_x2(e); F’_y2(e); M_2(e)*m^-1]*kN^-1
#novar
#for e = 1 : n_E
    F_E(e)
#loop
#varsub
'<h4>Global load vector</h4>
#hide
F = vector(3*n_J)
#for i = 1 : n_q
    e = q.(i; 1)
    #for jj = 1 : 2
        j = 3*e_J.(e; jj) - 3
        F.(j + 1) = F.(j + 1) + take(3*jj - 2; F_E(e))
        F.(j + 2) = F.(j + 2) + take(3*jj - 1; F_E(e))
        F.(j + 3) = F.(j + 3) + take(3*jj; F_E(e))
    #loop
#loop
#show
F
'<h4>Results</h4>
'<p><strong>Solution of the system of equations</strong></p>
Z = clsolve(K; F)
'<p><strong>Joint displacements</strong></p>
z_J(j) = slice(Z; 3*j - 2; 3*j)
z(j) = round(z_J(j)/δz)*δz*1000*[mm; mm; 1]
#novar
#for j = 1 : n_J
    z(j)
#loop
#varsub
'<p><strong>Support reactions</strong></p>
r(i) = row(c; i)','j(i) = take(1; r(i))
R(i) = -z_J(j(i))*[m; m; 1]*last(r(i); 3)
#novar
#for i = 1 : n_c
    #val
    '<p>Joint <b>J'j(i)' -
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    #equ
    '</b>'R(i)'</p>
#loop
#varsub
'<p><strong>Element end forces</strong></p>
z_E(e) = [z_J(e_J.(e; 1)); z_J(e_J.(e; 2))]
R_E(e) = col(k_E(e)*T(e)*z_E(e) - T(e)*F_E(e); 1)*[1; 1; m; 1; 1; m]*kN
#novar
#for e = 1 : n_E
    R_E(e)
#loop
#varsub
'<p><strong>Element internal forces</strong></p>
N(e; x) = -take(1; R_E(e)) + N_0(e; x)
Q(e; x) = take(2; R_E(e)) + Q_0(e; x)
M(e; x) = -take(3; R_E(e)) + take(2; R_E(e))*x + M_0(e; x)
#hide
w = max(x_J)
h = max(y_J)
W = 240
H = h*W/w
k = W/w
#def red_style$ = style = "stroke:red; stroke-width:1; fill:red"
#deg
#for i = 1 : 3
    #hide
    R(e; x) = take(i; N(e; x); Q(e; x); M(e; x))
    sgn = take(i; 1; 1; -1)
    tol = 0.01*take(i; kN; kN; kNm)
    R_max = $Sup{$Sup{R(e; x) @ x = 0m : l(e)} @ e = 1 : n_E}
    R_min = $Sup{abs($Inf{R(e; x) @ x = 0m : l(e)}) @ e = 1 : n_E}
    k_R = sgn*1m*k/max(R_min; R_max)
    #show
    #if i ≡ 1
        '<p><strong>Axial forces diagram, kN</strong></p>
    #else if i ≡ 2
        '<p><strong>Shear forces diagram, kN</strong></p>
    #else
        '<p><strong>Bending moments diagram, kNm</strong></p>
    #end if
    #val
    svg$
    '<use href="#frame"/>
    #for e = 1 : n_E
        #hide
        x1 = x_1(e)*k
        y1 = (h - y_1(e))*k
        x2 = x_2(e)*k
        y2 = (h - y_2(e))*k
        c_e = c(e)
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```
        s_e = s(e)
        l_e = l(e)
        st = l_e/10
        xd2 = x1
        yd2 = y1
        #show
        #for j = 0 : 10
            #hide
            xd1 = xd2
            yd1 = yd2
            x = j*st*k
            v = R(e; j*st)
            y = v*k_R
            xd2 = x1 + x*c_e - y*s_e
            yd2 = y1 - x*s_e - y*c_e
            α = 90 + atan2(c_e; s_e)
            #if α ≥ 135
                α = α - 180
            #end if
            #if α < -45
                α = α + 180
            #else if α < 0
                α = 360 + α
            #end if
            d = -15*sign(v*sgn)
            #show
            line$(xd1; yd1; xd2; yd2; red_style$)
            #if (j ≡ 0 ∨ j ≡ 10) ∧ abs(v) > tol
                text$(xd2 + s_e*d; yd2 + d*c_e; α; 'v')
            #end if
            line$(xd1; yd1; xd2; yd2; red_style$)
        #loop
        #hide
        xd1 = x2
        yd1 = y2
        #show
        line$(xd1; yd1; xd2; yd2; red_style$)
    #loop
    '</svg>
#loop
#equ
'<p><strong>Deformed shape</strong></p>
'Shape function in relative coordinates ξ = x/l (approximate)
Φ_1(e; ξ) = 1 - 3*ξ^2 + 2*ξ^3
Φ_2(e; ξ) = ξ*l(e)*m^-1*(1 - 2*ξ + ξ^2)
Φ_3(e; ξ) = ξ^2*(3 - 2*ξ)
Φ_4(e; ξ) = ξ^2*l(e)*m^-1*(-1 + ξ)
'Element endpoint displacements and rotations
z_E,loc(e) = T(e)*z_E(e)
u_1(e) = take(1; z_E,loc(e))','v_1(e) = take(2; z_E,loc(e))','ϕ_1(e) =
```

```
take(3; z_E,loc(e))
u_2(e) = take(4; z_E,loc(e))','v_2(e) = take(5; z_E,loc(e))','ϕ_2(e) = take(6; z_E,loc(e))
'Displacement functions
u(e; ξ) = u_1(e)*(1 - ξ) + u_2(e)*ξ
v(e; ξ) = v_1(e)*Φ_1(e; ξ) + ϕ_1(e)*Φ_2(e; ξ) + v_2(e)*Φ_3(e; ξ) + ϕ_2(e)*Φ_4(e; ξ)
'Deformed shape, mm
#val
#hide
tol = 0.00001
k_R = 800
#show
svg$
'<use href="#frame" style="opacity:0.4;"/>
#for e = 1 : n_E
    #hide
    x1 = x_1(e)*k
    y1 = (h - y_1(e))*k
    x2 = x_2(e)*k
    y2 = (h - y_2(e))*k
    c_e = c(e)
    s_e = s(e)
    l_e = l(e)
    u = u(e; 0)
    v = v(e; 0)
    x = u*k_R
    y = v*k_R
    xd2 = x1 + x*c_e - y*s_e
    yd2 = y1 - x*s_e - y*c_e
    #show
    #for j = 0 : 10
        #hide
        xd1 = xd2
        yd1 = yd2
        ξ = j/10
        u = u(e; ξ)
        v = v(e; ξ)
        x = ξ*l_e*k + u*k_R
        y = v*k_R
        xd2 = x1 + x*c_e - y*s_e
        yd2 = y1 - x*s_e - y*c_e
        d = -15*sign(v)
        #show
        line$(xd1; yd1; xd2; yd2; red_style$)
    #loop
#loop
#for j = 1 : n_J
    #hide
    z_J = z_J(j)
```

```
    u = z_J.1
    v = z_J.2
    x = x_J.j*k + u*k_R
    y = (h - y_J.j)*k - v*k_R
    dx = 15*sign(u)
    dy = -15*sign(v)
    #show
    #if abs(u) > tol
        texth$(x + dx; y; 'u*1000')
    #end if
    #if abs(v) > tol
        textv$(x; y + dy; 'v*1000')
    #end if
#loop
'</svg>
#equ
```

III. Output

Analysis of plane frames with variable cross-sections

Joint coordinates

$$\begin{array}{lccccc} & \text{J1} & \text{J2} & \text{J3} & \text{J4} & \text{J5} \\ \vec{x}_J = & [0\,\text{m} & 0\,\text{m} & 8\,\text{m} & 16\,\text{m} & 16\,\text{m}] \\ \vec{y}_J = & [0\,\text{m} & 8\,\text{m} & 10\,\text{m} & 8\,\text{m} & 0\,\text{m}] \end{array}$$

$$n_J = \mathbf{len}\left(\vec{x}_J\right) = 5$$

Elements

$$e_J^T = \begin{bmatrix} 1 & 3 & 3 & 5 \\ 2 & 2 & 4 & 4 \end{bmatrix}$$

$$n_E = \mathbf{n}_{rows}\left(e_J\right) = 4$$

Element endpoint coordinates

$$x_1(e) = \vec{x}_{J.e_{J.e,1}},\ y_1(e) = \vec{y}_{J.e_{J.e,1}},\ x_2(e) = \vec{x}_{J.e_{J.e,2}},\ y_2(e) = \vec{y}_{J.e_{J.e,2}}$$

Element length- $l(e) = \sqrt{\left(x_2(e) - x_1(e)\right)^2 + \left(y_2(e) - y_1(e)\right)^2}$

Element directions - $c(e) = \dfrac{x_2(e) - x_1(e)}{l(e)}$, $s(e) = \dfrac{y_2(e) - y_1(e)}{l(e)}$

Transformation matrix

Diagonal 3x3 block - $t(e) = [c(e);\ s(e);\ 0 \mid -s(e);\ c(e);\ 0 \mid 0;\ 0;\ 1]$

Generation of the full transformation matrix

$$T(e) = \mathbf{add}\left(t(e);\ \mathbf{add}\left(t(e);\ \mathbf{matrix}(6;\ 6);\ 1;\ 1\right);\ 4;\ 4\right)$$

Supports

$$c=\begin{array}{c}\begin{array}{cccc} J & cx & cy & cr \end{array}\\ \begin{bmatrix} 1 & 10^{20}\,\mathrm{kN/m} & 10^{20}\,\mathrm{kN/m} & 0\,\mathrm{kNm} \\ 5 & 10^{20}\,\mathrm{kN/m} & 10^{20}\,\mathrm{kN/m} & 10^{20}\,\mathrm{kNm} \end{bmatrix}\end{array}$$

$n_c = \mathrm{n}_{rows}(c) = 2$

Loads

$$q=\begin{array}{c}\begin{array}{ccc} e & qx & qy \end{array}\\ \begin{bmatrix} 1 & 10\,\mathrm{kN/m} & 0\,\mathrm{kN/m} \\ 2 & 0\,\mathrm{kN/m} & -20\,\mathrm{kN/m} \\ 3 & 0\,\mathrm{kN/m} & -10\,\mathrm{kN/m} \end{bmatrix}\end{array}$$

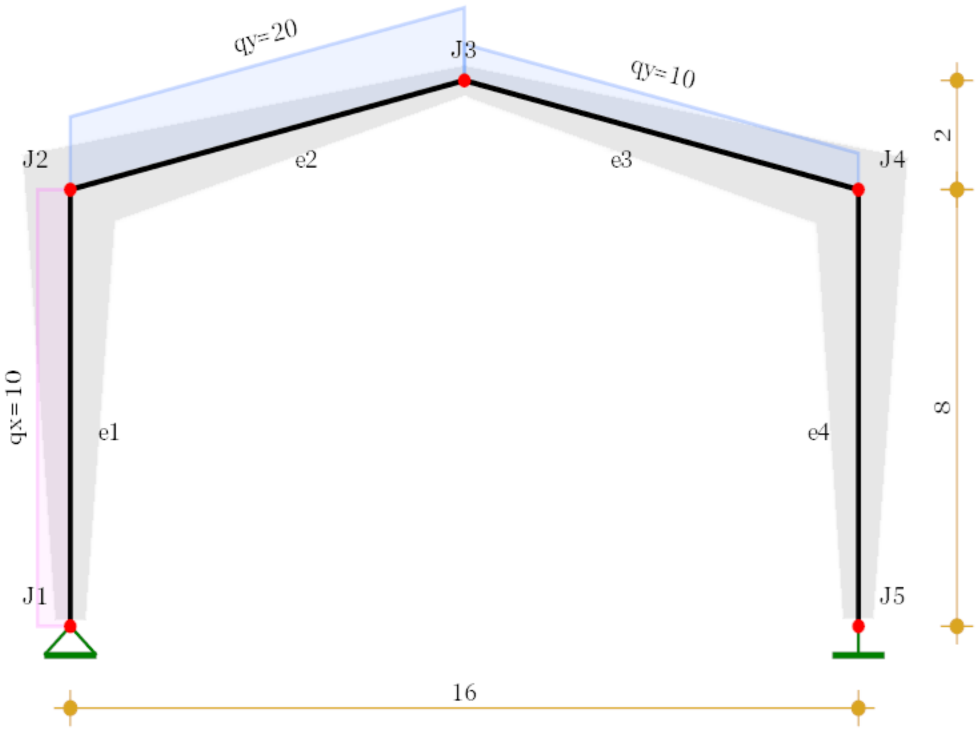
Load values on elements

$$\begin{array}{ccccc} & E1 & E2 & E3 & E4 \end{array}$$

$\vec{q}_x = [10\ \mathrm{kN/m} \quad 0\ \mathrm{kN/m} \quad 0\ \mathrm{kN/m} \quad 0\ \mathrm{kN/m}]$

$\vec{q}_y = [0\ \mathrm{kN/m} \quad -20\ \mathrm{kN/m} \quad -10\ \mathrm{kN/m} \quad 0\ \mathrm{kN/m}]$

Scheme of the structure



Materials

Modules of elasticity - $\vec{E} = [45\ \mathrm{GPa} \quad 35\ \mathrm{GPa}]$

Poisson coefficients - $\vec{\nu} = [0.2 \quad 0.2]$

Shear modules - $\vec{G} = \dfrac{\vec{E}}{2 \cdot (1+\vec{\nu})} = [18.75\ \mathrm{GPa} \quad 14.58\ \mathrm{GPa}]$

Assignment on elements - $\vec{e}_M = [1 \quad 2 \quad 2 \quad 1]$

Cross-sections

Section S1 - $b_1 = 300\ \mathrm{mm}$, $h_1 = 300\ \mathrm{mm}$

Section S2 - $b_2 = 300\ \text{mm}$, $h_2 = 900\ \text{mm}$

General representation

Width - $b(\xi;\ z) = b_1 + (b_2 - b_1) \cdot \xi$

Height - $h(\xi) = h_1 + (h_2 - h_1) \cdot \xi$

Cross section properties

Area - $$A(\xi) = \int_{0\ \text{mm}}^{h(\xi)} b(\xi;\ z)\, \mathrm{d}z$$

First moment of area - $$S(\xi) = \int_{0\ \text{mm}}^{h(\xi)} b(\xi;\ z) \cdot z\, \mathrm{d}z$$

Centroid - $$z_c(\xi) = \frac{S(\xi)}{A(\xi)}$$

Second moment of area - $$I(\xi) = \int_{0\ \text{mm}}^{h(\xi)} b(\xi;\ z) \cdot \left(z - z_c(\xi)\right)^2 \mathrm{d}z$$

First moment of area bellow z - $$S_1(\xi;\ z) = \int_{0\ \text{mm}}^{z} b(\xi;\ \zeta) \cdot \left(z_c(\xi) - \zeta\right) \mathrm{d}\zeta$$

Shear area - $$A_Q(\xi) = \frac{I(\xi)^2}{\displaystyle\int_{0\ \text{mm}}^{h(\xi)} \frac{S_1(\xi;\ z)^2}{b(\xi;\ z)}\, \mathrm{d}z}$$

Element stiffness matrix

Elastic properties for element "e"

$$EA(e;\ x) = \vec{E}_{\vec{e}_{M.e}} \cdot A\left(\frac{x}{l(e)}\right), EI(e;\ x) = \vec{E}_{\vec{e}_{M.e}} \cdot I\left(\frac{x}{l(e)}\right), GA_Q(e;\ x) = \vec{G}_{\vec{e}_{M.e}} \cdot A_Q\left(\frac{x}{l(e)}\right)$$

Stiffness matrix for element with variable cross-section

Displacement due to F_x = 1 in primary system - $$u_F(e) = \int_{0\ \text{m}}^{l(e)} \frac{1}{EA(e;\ x)}\, \mathrm{d}x$$

Displacement due to F_{y1} = 1 in primary system, with account for shear deflections - $$v_{F1}(e) = \int_{0\ \text{m}}^{l(e)} \frac{x^2}{EI(e;\ x)}\, \mathrm{d}x + \int_{0\ \text{m}}^{l(e)} \frac{1}{GA_Q(e;\ x)}\, \mathrm{d}x$$

Rotation due to F_{y1} = 1 in primary system - $$\varphi_{F1}(e) = -\left(\int_{0\ \text{m}}^{l(e)} \frac{x}{EI(e;\ x)}\, \mathrm{d}x\right)$$

Displacement due to M_1 = 1 in primary system - $v_{M1}(e) = \varphi_{F1}(e)$

Rotation due to M_1 = 1 in primary system - $\varphi_{M1}(e) = \int_{0\,\mathrm{m}}^{l(e)} \frac{1}{EI(e;\ x)}\,\mathrm{d}x$

Determinant - $D_1(e) = \varphi_{M1}(e)\cdot v_{F1}(e) - \varphi_{F1}(e)^2$

Displacement due to F_{y2} = 1 in primary system - $v_{F2}(e) = \int_{0\,\mathrm{m}}^{l(e)} \frac{(l(e)-x)^2}{EI(e;\ x)}\,\mathrm{d}x + \int_{0\,\mathrm{m}}^{l(e)} \frac{1}{GA_Q(e;\ x)}\,\mathrm{d}x$

Rotation due to F_{y2} = 1 in primary system - $\varphi_{F2}(e) = \int_{0\,\mathrm{m}}^{l(e)} \frac{l(e)-x}{EI(e;\ x)}\,\mathrm{d}x$

Displacement due to M_2 = 1 in primary system - $v_{M2}(e) = \varphi_{F2}(e)$

Rotation due to M_2 = 1 in primary system - $\varphi_{M2}(e) = \varphi_{M1}(e)$

Determinant - $D_2(e) = \varphi_{M2}(e)\cdot v_{F2}(e) - \varphi_{F2}(e)^2$

3x3 blocks of the stiffness matrix for element "e"

$$k_{ii}(e) = \left[\frac{D_1(e)}{u_F(e)}\cdot kNm\,|\,0;\ \varphi_{M1}(e)\cdot kNm;\ -\varphi_{F1}(e)\cdot kN\,|\,0;\ -\varphi_{F1}(e)\cdot kN;\ v_{F1}(e)\cdot\frac{kN}{m}\right]\cdot\frac{kN^{-2}}{D_1(e)}$$

$$k_{ij}(e) = \left[-\frac{D_2(e)}{u_F(e)}\cdot kNm\,|\,0;\ -\varphi_{M2}(e)\cdot kNm;\ \varphi_{F2}(e)\cdot kN\,|\,0;\ \left(\varphi_{F2}(e)-\varphi_{M2}(e)\cdot l(e)\right)\cdot kN;\ -\left(v_{F2}(e)-\varphi_{F2}(e)\cdot l(e)\right)\cdot\frac{kN}{m}\right]\cdot\frac{kN^{-2}}{D_2(e)}$$

$$k_{ji}(e) = \left[-\frac{D_1(e)}{u_F(e)}\cdot kNm\,|\,0;\ -\varphi_{M1}(e)\cdot kNm;\ \varphi_{F1}(e)\cdot kN\,|\,0;\ \left(\varphi_{F1}(e)+\varphi_{M1}(e)\cdot l(e)\right)\cdot kN;\ -\left(v_{F1}(e)+\varphi_{F1}(e)\cdot l(e)\right)\cdot\frac{kN}{m}\right]\cdot\frac{kN^{-2}}{D_1(e)}$$

$$k_{jj}(e) = \left[\frac{D_2(e)}{u_F(e)}\cdot kNm\,|\,0;\ \varphi_{M2}(e)\cdot kNm;\ -\varphi_{F2}(e)\cdot kN\,|\,0;\ -\varphi_{F2}(e)\cdot kN;\ v_{F2}(e)\cdot\frac{kN}{m}\right]\cdot\frac{kN^{-2}}{D_2(e)}$$

Full element stiffness matrix

$$k_E(e) = \mathbf{stack}\left(\mathbf{augment}\left(k_{ii}(e);\ k_{ij}(e)\right);\ \mathbf{augment}\left(k_{ji}(e);\ k_{jj}(e)\right)\right)$$

Stiffness matrices obtained in local coordinates

$$k_E(1) = \begin{bmatrix} 921617 & 0 & 0 & -921617 & 0 & 0 \\ 0 & 4741.71 & 9483.42 & 0 & -4741.71 & 28450.3 \\ 0 & 9483.42 & 36052.8 & 0 & -9483.42 & 39814.6 \\ -921617 & 0 & 0 & 921617 & 0 & 0 \\ 0 & -4741.71 & -9483.42 & 0 & 4741.71 & -28450.3 \\ 0 & 28450.3 & 39814.6 & 0 & -28450.3 & 187788 \end{bmatrix}$$

$$k_E(2)=\begin{bmatrix} 695411 & 0 & 0 & -695411 & 0 & 0 \\ 0 & 3370.35 & 6948.16 & 0 & -3370.35 & 20844.5 \\ 0 & 6948.16 & 27216.3 & 0 & -6948.16 & 30079.7 \\ -695411 & 0 & 0 & 695411 & 0 & 0 \\ 0 & -3370.35 & -6948.16 & 0 & 3370.35 & -20844.5 \\ 0 & 20844.5 & 30079.7 & 0 & -20844.5 & 141808 \end{bmatrix}$$

Stiffness matrices obtained in global coordinates

$$\mathbf{transp}(T(1))\cdot k_E(1)\cdot T(1)=\begin{bmatrix} 4741.71 & 0 & -9483.42 & -4741.71 & 0 & -28450.3 \\ 0 & 921617 & 0 & 0 & -921617 & 0 \\ -9483.42 & 0 & 36052.8 & 9483.42 & 0 & 39814.6 \\ -4741.71 & 0 & 9483.42 & 4741.71 & 0 & 28450.3 \\ 0 & -921617 & 0 & 0 & 921617 & 0 \\ -28450.3 & 0 & 39814.6 & 28450.3 & 0 & 187788 \end{bmatrix}$$

$$\mathbf{transp}(T(2))\cdot k_E(2)\cdot T(2)=\begin{bmatrix} 654703 & 162833 & 1685.18 & -654703 & -162833 & 5055.53 \\ 162833 & 44078.6 & -6740.7 & -162833 & -44078.6 & -20222.1 \\ 1685.18 & -6740.7 & 27216.3 & -1685.18 & 6740.7 & 30079.7 \\ -654703 & -162833 & -1685.18 & 654703 & 162833 & -5055.53 \\ -162833 & -44078.6 & 6740.7 & 162833 & 44078.6 & 20222.1 \\ 5055.53 & -20222.1 & 30079.7 & -5055.53 & 20222.1 & 141808 \end{bmatrix}$$

Global stiffness matrix

$$K=\begin{bmatrix} 10^{20} & 0 & -9483.42 & -4741.71 & 0 & -28450.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^{20} & 0 & 0 & -921617 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9483.42 & 0 & 36052.8 & 9483.42 & 0 & 39814.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4741.71 & 0 & 9483.42 & 659445 & 162833 & 23394.7 & -654703 & -162833 & -1685.18 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -921617 & 0 & 162833 & 965696 & 20222.1 & -162833 & -44078.6 & 6740.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -28450.3 & 0 & 39814.6 & 23394.7 & 20222.1 & 329596 & 5055.53 & -20222.1 & 30079.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -654703 & -162833 & 5055.53 & 1309406 & 0 & 3370.35 & -654703 & 162833 & 5055.53 & 0 & 0 & 0 \\ 0 & 0 & 0 & -162833 & -44078.6 & -20222.1 & 0 & 88157.3 & 0 & 162833 & -44078.6 & 20222.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1685.18 & 6740.7 & 30079.7 & 3370.35 & 0 & 54432.6 & -1685.18 & -6740.7 & 30079.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -654703 & 162833 & -1685.18 & 659445 & -162833 & 23394.7 & -4741.71 & 0 & 9483.42 \\ 0 & 0 & 0 & 0 & 0 & 0 & 162833 & -44078.6 & -6740.7 & -162833 & 965696 & -20222.1 & 0 & -921617 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5055.53 & 20222.1 & 30079.7 & 23394.7 & -20222.1 & 329596 & -28450.3 & 0 & 39814.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4741.71 & 0 & -28450.3 & 10^{20} & 0 & -9483.42 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -921617 & 0 & 0 & 10^{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9483.42 & 0 & 39814.6 & -9483.42 & 0 & 10^{20} \end{bmatrix}$$

Element load vector

Load functions

Lateral - $q_E(e;\ x)=-\vec{q}_{x.e}\cdot s(e)+\vec{q}_{y.e}\cdot c(e)$

Axial - $n_E(e;\ x)=\vec{q}_{x.e}\cdot c(e)+\vec{q}_{y.e}\cdot s(e)$

Functions of internal forces in primary system

Axial forces - $N_0(e;\ x)=-\int_{0\,\mathrm{m}}^{x} n_E(e;\ \xi)\,\mathrm{d}\xi$

Shear forces - $Q_0(e;\ x)=\int_{0\,\mathrm{m}}^{x} q_E(e;\ \xi)\,\mathrm{d}\xi$

Bending moment - $M_0(e;\ x)=\int\limits_{0\,\mathrm{m}}^{x} q_E(e;\ \xi)\cdot(x-\xi)\,\mathrm{d}\xi$

Reactions at element ends

Displacements along “x“ due to axial loads - $u_n(e)=\int\limits_{0\,\mathrm{m}}^{l(e)} \frac{N_0(e;\ x)}{EA(e;\ x)}\,\mathrm{d}x$

Displacements along “y“ due to lateral loads - $v_q(e)=\int\limits_{0\,\mathrm{m}}^{l(e)} \frac{M_0(e;\ x)\cdot x}{EI(e;\ x)}\,\mathrm{d}x+\int\limits_{0\,\mathrm{m}}^{l(e)} \frac{Q_0(e;\ x)}{GA_Q(e;\ x)}\,\mathrm{d}x$

Rotations due to lateral loads - $\varphi_q(e)=-\int\limits_{0\,\mathrm{m}}^{l(e)} \frac{M_0(e;\ x)}{EI(e;\ x)}\,\mathrm{d}x$

Element endpoint loads in local coordinate system

For joint “1”:

$$F_{x1}(e)=-\frac{u_n(e)}{u_F(e)}$$

$$F_{y1}(e)=\frac{\varphi_{M1}(e)\cdot v_q(e)-\varphi_{F1}(e)\cdot\varphi_q(e)}{D_1(e)}$$

$$M_1(e)=\frac{v_{F1}(e)\cdot\varphi_q(e)-\varphi_{F1}(e)\cdot v_q(e)}{D_1(e)}$$

For joint “2”:

$$F_{x2}(e)=-F_{x1}(e)-N_0(e;l(e))$$

$$F_{y2}(e)=-F_{y1}(e)+Q_0(e;l(e))$$

$$M_2(e)=-M_1(e)+F_{y1}(e)\cdot l(e)-M_0(e;l(e))$$

Element endpoint loads in global coordinate system

For joint “1”:

$$F'_{x1}(e)=F_{x1}(e)\cdot c(e)-F_{y1}(e)\cdot s(e)$$

$$F'_{y1}(e)=F_{x1}(e)\cdot s(e)+F_{y1}(e)\cdot c(e)$$

For joint “2”:

$$F'_{x2}(e)=F_{x2}(e)\cdot c(e)-F_{y2}(e)\cdot s(e)$$

$$F'_{y2}(e)=F_{x2}(e)\cdot s(e)+F_{y2}(e)\cdot c(e)$$

Element load vector

$$F_E(e)=\left[F'_{x1}(e);\ F'_{y1}(e);\ M_1(e)\cdot m^{-1};\ F'_{x2}(e);\ F'_{y2}(e);\ M_2(e)\cdot m^{-1}\right]\cdot kN^{-1}$$

$$F_E(1)=[31.43\quad 0\quad -25.11\quad 48.57\quad 0\quad 93.68]$$

$$F_E(2)=[-0.675\quad -64.96\quad 51.75\quad 0.675\quad -99.97\quad -193.14]$$

$$F_E(3)=[0.338\quad -32.48\quad -25.88\quad -0.338\quad -49.98\quad 96.57]$$

$$F_E(4)=[0\quad 0\quad 0\quad 0\quad 0\quad 0]$$

Global load vector

$$\vec{F}=\begin{bmatrix}31.43 & 0 & -25.11 & 49.25 & -99.97 & -99.46 & -0.338\\ -97.44 & 25.88 & -0.338 & -49.98 & 96.57 & 0 & 0 & 0\end{bmatrix}$$

Results

Solution of the system of equations

$$\vec{Z} = \text{clsolve}\left(K\,;\ \vec{F}\right)$$

$$\vec{Z} = [\,1.06 \times 10^{-19} \quad -1.34 \times 10^{-18} \quad -0.00122 \quad 0.0112 \quad -0.000145 \quad -0.0022 \quad 0.0146 \quad -0.0139 \quad 0.00199 \quad 0.0179 \quad -0.000124 \quad -0.000536 \quad 6.94 \times 10^{-19} \quad -1.14 \times 10^{-18} \quad -1.48 \times 10^{-18}\,]$$

Joint displacements

$$z_J(j) = \textbf{slice}\left(\vec{Z}\,;\ 3 \cdot j - 2\,;\ 3 \cdot j\right)$$

$$z(j) = \textbf{round}\left(\frac{z_J(j)}{\delta z}\right) \cdot \delta z \cdot 1000 \cdot [\text{mm; mm; } 1]$$

$$z(1) = [0\ \text{mm} \quad 0\ \text{mm} \quad -1.22]$$

$$z(2) = [11.23\ \text{mm} \quad -0.145\ \text{mm} \quad -2.2]$$

$$z(3) = [14.55\ \text{mm} \quad -13.87\ \text{mm} \quad 1.99]$$

$$z(4) = [17.86\ \text{mm} \quad -0.124\ \text{mm} \quad -0.536]$$

$$z(5) = [0\ \text{mm} \quad 0\ \text{mm} \quad 0]$$

Support reactions

$$r(i) = \textbf{row}(c\,;\ i)\ ,\ j(i) = \textbf{take}\left(1\,;\ r(i)\right)$$

$$R(i) = -z_J\left(j(i)\right) \cdot [\text{m; m; } 1] \cdot \textbf{last}\left(r(i)\,;\ 3\right)$$

Joint **J1** - $R(1) = [-10.56\ \text{kN} \quad 133.56\ \text{kN} \quad 0\ \text{kNm}]$

Joint **J5** - $R(2) = [-69.44\ \text{kN} \quad 113.82\ \text{kN} \quad 148.05\ \text{kNm}]$

Element end forces

$$z_E(e) = \left[z_J\left(e_{J.e,1}\right);\ z_J\left(e_{J.e,2}\right)\right]$$

$$R_E(e) = \textbf{col}\left(k_E(e) \cdot T(e) \cdot z_E(e) - T(e) \cdot F_E(e)\,;\ 1\right) \cdot [1;\ 1;\ \text{m};\ 1;\ 1;\ \text{m}] \cdot \text{kN}$$

$$R_E(1) = [133.56\ \text{kN} \quad 10.56\ \text{kN} \quad 1.78 \times 10^{-14}\ \text{kNm} \quad -133.56\ \text{kN} \quad 69.44\ \text{kN} \quad -235.56\ \text{kNm}]$$

$$R_E(2) = [59.76\ \text{kN} \quad -47.27\ \text{kN} \quad 34.36\ \text{kNm} \quad -99.76\ \text{kN} \quad -112.73\ \text{kN} \quad 235.56\ \text{kNm}]$$

$$R_E(3) = [74.98\ \text{kN} \quad -13.58\ \text{kN} \quad -34.36\ \text{kNm} \quad -94.98\ \text{kN} \quad 93.58\ \text{kN} \quad -407.5\ \text{kNm}]$$

$$R_E(4) = [113.82\ \text{kN} \quad 69.44\ \text{kN} \quad 148.05\ \text{kNm} \quad -113.82\ \text{kN} \quad -69.44\ \text{kN} \quad 407.5\ \text{kNm}]$$

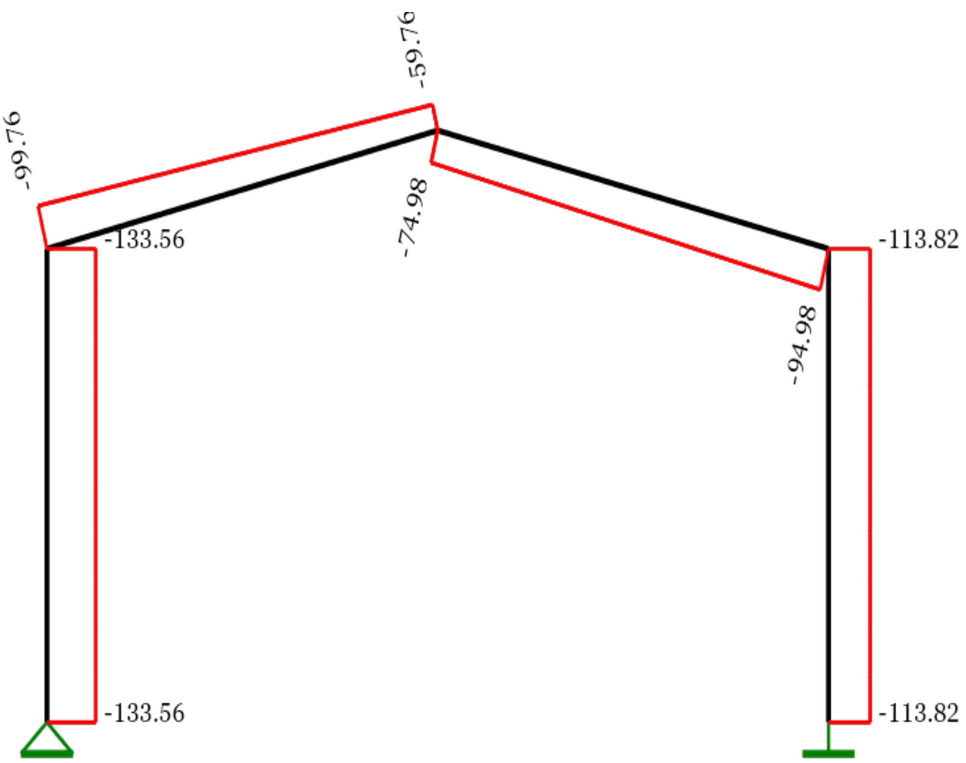
Element internal forces

$$N(e;\ x) = \textbf{-take}\left(1\,;\ R_E(e)\right) + N_0(e;\ x)$$

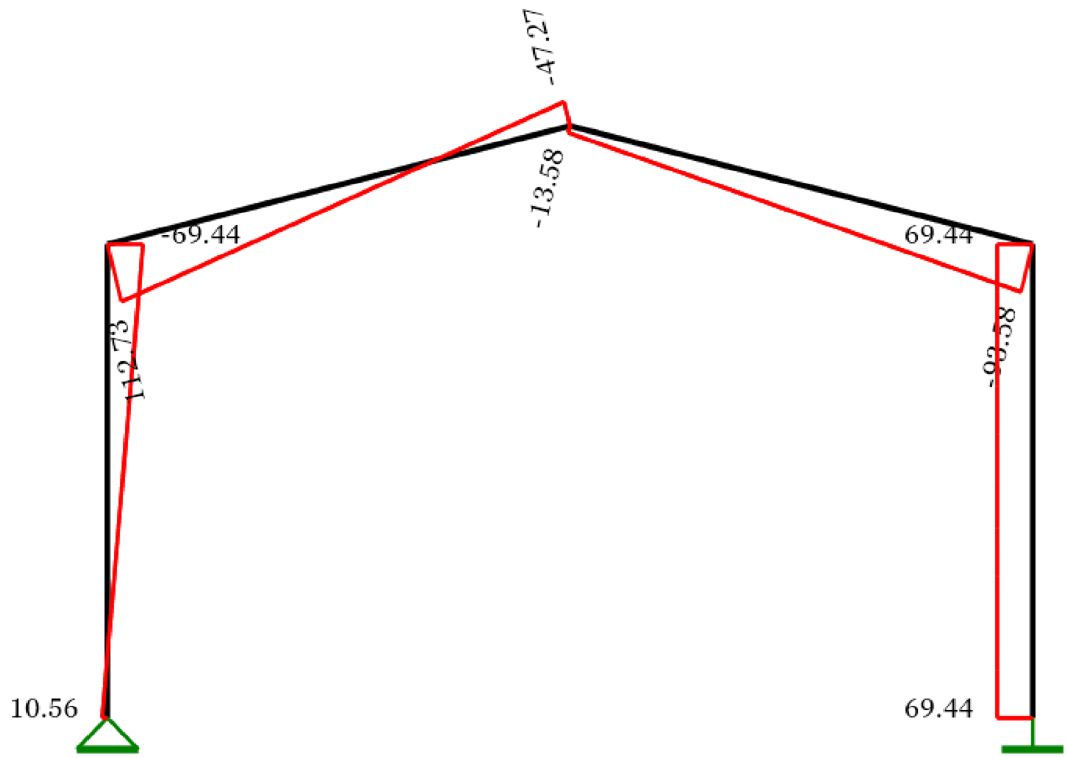
$$Q(e;\ x) = \textbf{take}\left(2\,;\ R_E(e)\right) + Q_0(e;\ x)$$

$$M(e;\ x) = \textbf{-take}(3;\ R_E(e)) + \textbf{take}(2;\ R_E(e)) \cdot x + M_0(e;\ x)$$

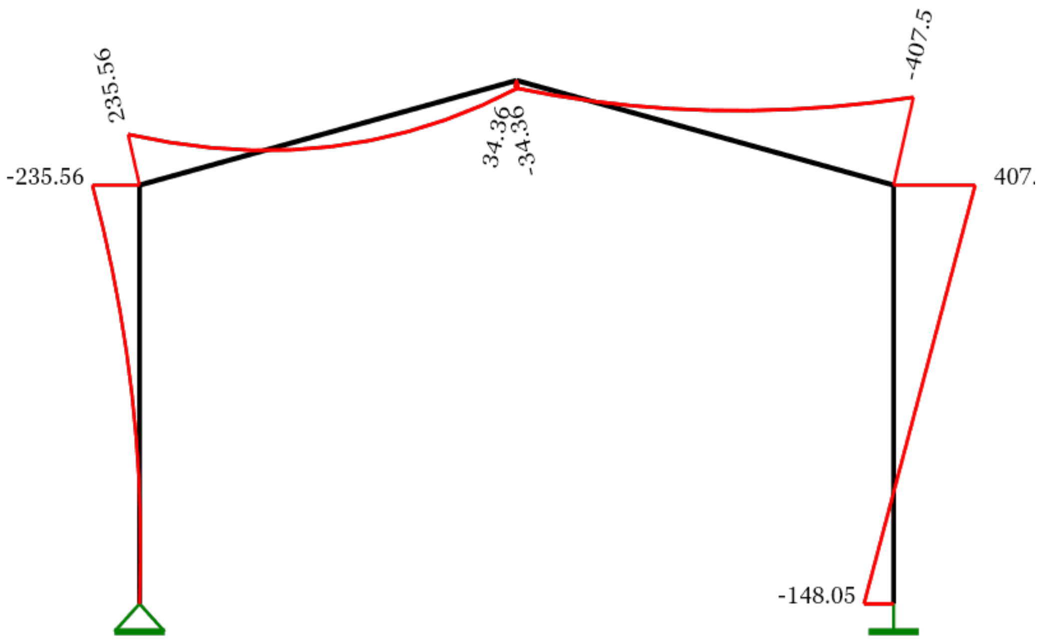
Axial forces diagram, kN



Shear forces diagram, kN



Bending moments diagram, kNm



Deformed shape

Shape function in relative coordinates ξ = x/l (approximate)

$$\Phi_1(e;\ \xi) = 1 - 3 \cdot \xi^2 + 2 \cdot \xi^3, \qquad \Phi_2(e;\ \xi) = \left(1 - 2 \cdot \xi + \xi^2\right) \cdot \xi \cdot \frac{l(e)}{m}$$

$$\Phi_3(e;\ \xi) = (3 - 2 \cdot \xi) \cdot \xi^2, \qquad \Phi_4(e;\ \xi) = (-1 + \xi) \cdot \xi^2 \cdot \frac{l(e)}{m}$$

Element endpoint displacements and rotations

$$z_{E,loc}(e) = T(e) \cdot z_E(e)$$

$$u_1(e) = \mathbf{take}\left(1;\ z_{E,loc}(e)\right),\quad v_1(e) = \mathbf{take}\left(2;\ z_{E,loc}(e)\right),\quad \varphi_1(e) = \mathbf{take}\left(3;\ z_{E,loc}(e)\right)$$

$$u_2(e) = \mathbf{take}\left(4;\ z_{E,loc}(e)\right),\quad v_2(e) = \mathbf{take}\left(5;\ z_{E,loc}(e)\right),\quad \varphi_2(e) = \mathbf{take}\left(6;\ z_{E,loc}(e)\right)$$

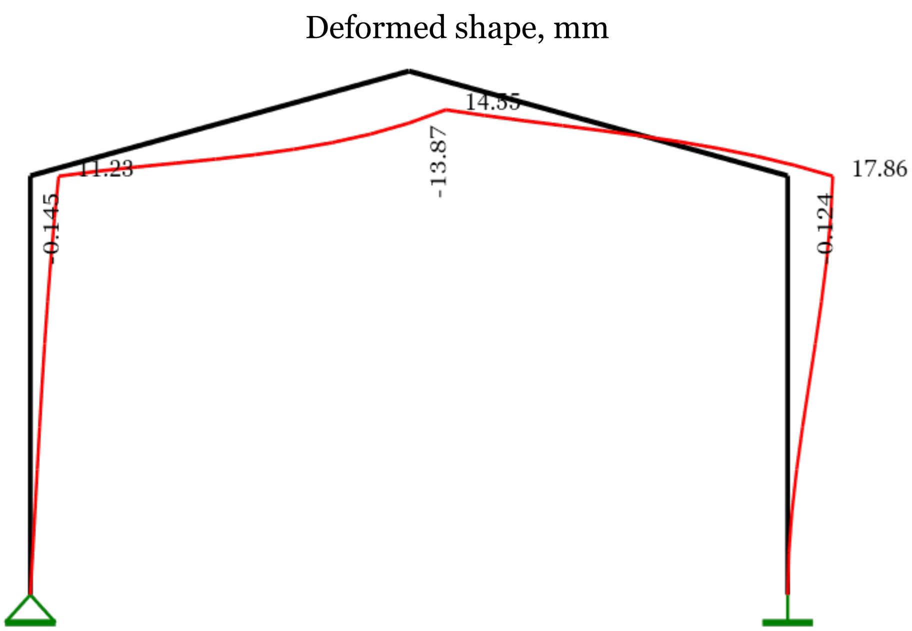
Displacement functions

$$x_m(e;\ \xi) = 0.5 \cdot \xi \cdot l(e)$$

$$u(e;\ \xi) = u_1(e) \cdot (1 - \xi) + u_2(e) \cdot \xi + \frac{n_E(e;\ x_m(e;\ \xi)) \cdot m}{EA(e;\ x_m(e;\ \xi))} \cdot \xi \cdot (1 - \xi)$$

$$v(e;\ \xi) = v_1(e) \cdot \Phi_1(e;\ \xi) + \varphi_1(e) \cdot \Phi_2(e;\ \xi) + v_2(e) \cdot \Phi_3(e;\ \xi) + \varphi_2(e) \cdot \Phi_4(e;\ \xi)$$

$$+ \frac{q_E(e;\ x_m(e;\ \xi)) \cdot l(e)^4}{24 \cdot EI(e;\ x_m(e;\ \xi))} \cdot \frac{\xi^2 \cdot (1 - \xi)^2}{\mathrm{m}} + \frac{q_E(e;\ x_m(e;\ \xi)) \cdot l(e)^2}{2 \cdot GA_s(e;\ x_m(e;\ \xi))} \cdot \frac{\xi \cdot (1 - \xi)}{\mathrm{m}}$$



IV. Comparison with SAP 2000

To verify the obtained results, a SAP 2000 model is developed for the same structure. A non-prismatic finite element with three intermediate points is used. The input and output data from the analysis is listed below in both text and graphics.

Input data:

S T A T I C   L O A D   C A S E S

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	DEAD	0.0000

J O I N T   D A T A

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS	ANGLE-A	ANGLE-B	ANGLE-C
1	-8.00000	0.00000	0.00000	1 1 1 0 0 0	0.000	0.000	0.000
2	-8.00000	0.00000	8.00000	0 0 0 0 0 0	0.000	0.000	0.000
3	8.00000	0.00000	0.00000	1 1 1 1 1 1	0.000	0.000	0.000
4	8.00000	0.00000	8.00000	0 0 0 0 0 0	0.000	0.000	0.000
5	0.00000	0.00000	10.00000	0 0 0 0 0 0	0.000	0.000	0.000

F R A M E   E L E M E N T   D A T A

FRAME	JNT-1	JNT-2	SECTION	ANGLE	RELEASES	SEGMENTS	R1	R2	FACTOR	LENGTH
1	1	2	VAR1	0.000	000000	2	0.000	0.000	1.000	8.000
2	3	4	VAR1	0.000	000000	2	0.000	0.000	1.000	8.000
3	5	2	VAR2	0.000	000000	2	0.000	0.000	1.000	8.246
4	5	4	VAR2	0.000	000000	2	0.000	0.000	1.000	8.246

M A T E R I A L   P R O P E R T Y   D A T A

MAT LABEL	MODULUS OF ELASTICITY	POISSON'S RATIO	THERMAL COEFF	WEIGHT PER UNIT VOL	MASS PER UNIT VOL
STEEL	199947979	0.300	1.170E-05	76.820	7.827
CONC	45000.000	0.200	9.900E-06	0.000	0.000
OTHER	24821128.4	0.200	9.900E-06	23.562	2.401
MAT1	45000.000	0.200	1.170E-05	0.000	0.000
MAT2	35000.000	0.300	1.170E-05	0.000	0.000

F R A M E   S E C T I O N   P R O P E R T Y   D A T A

SECTION LABEL	MAT LABEL	SECTION TYPE	DEPTH	FLANGE WIDTH TOP	FLANGE THICK TOP	WEB THICK	FLANGE WIDTH BOTTOM	FLANGE THICK BOTTOM
VAR1								
FSEC10	MAT1		0.300	0.300	0.000	0.000	0.000	0.000
FSEC11	MAT1		0.600	0.300	0.000	0.000	0.000	0.000
FSEC12	MAT1		0.900	0.300	0.000	0.000	0.000	0.000
VAR2								
FSEC20	MAT2		0.300	0.300	0.000	0.000	0.000	0.000
FSEC21	MAT2		0.600	0.300	0.000	0.000	0.000	0.000
FSEC22	MAT2		0.900	0.300	0.000	0.000	0.000	0.000

F R A M E   S E C T I O N   P R O P E R T Y   D A T A

SECTION LABEL	AREA	TORSIONAL INERTIA	MOMENTS OF INERTIA I33	I22	SHEAR AREAS A2	A3
FSEC10	9.000E-02	1.141E-03	6.750E-04	6.750E-04	7.500E-02	7.500E-02
FSEC11	0.180	3.708E-03	5.400E-03	1.350E-03	0.150	0.150
FSEC12	0.270	6.401E-03	1.823E-02	2.025E-03	0.225	0.225
FSEC20	9.000E-02	1.141E-03	6.750E-04	6.750E-04	7.500E-02	7.500E-02
FSEC22	0.270	6.401E-03	1.823E-02	2.025E-03	0.225	0.225
FSEC21	0.180	3.708E-03	5.400E-03	1.350E-03	0.150	0.150

F R A M E   S E C T I O N   P R O P E R T Y   D A T A

SECTION LABEL	S ECTION MODULII S33	S22	PLASTIC MODULII Z33	Z22	RADII OF GYRATION R33	R22
FSEC10	4.500E-03	4.500E-03	6.750E-03	6.750E-03	8.660E-02	8.660E-02
FSEC20	4.500E-03	4.500E-03	6.750E-03	6.750E-03	8.660E-02	8.660E-02
FSEC22	4.050E-02	1.350E-02	6.075E-02	2.025E-02	0.260	8.660E-02
FSEC11	1.800E-02	9.000E-03	2.700E-02	1.350E-02	0.173	8.660E-02
FSEC21	1.800E-02	9.000E-03	2.700E-02	1.350E-02	0.173	8.660E-02
FSEC12	4.050E-02	1.350E-02	6.075E-02	2.025E-02	0.260	8.660E-02

F R A M E   S P A N   D I S T R I B U T E D   L O A D S   Load Case LOAD1

FRAME	TYPE	DIRECTION	DISTANCE-A	VALUE-A	DISTANCE-B	VALUE-B
3	FORCE	GLOBAL-Z	0.0000	-20.0000	1.0000	-20.0000
4	FORCE	GLOBAL-Z	0.0000	-10.0000	1.0000	-10.0000
1	FORCE	GLOBAL-X	0.0000	10.0000	1.0000	10.0000

Results:

J O I N T   D I S P L A C E M E N T S

JOINT	LOAD	U1	U2	U3	R1	R2	R3
1	LOAD1	0	0	0	0	1.17	0
2	LOAD1	10.80	0	-0.15	0	2.13	0
3	LOAD1	0	0	0	0	0	0
4	LOAD1	17.26	0	-0.12	0	0.51	0
5	LOAD1	14.04	0	-13.53	0	-1.87	0

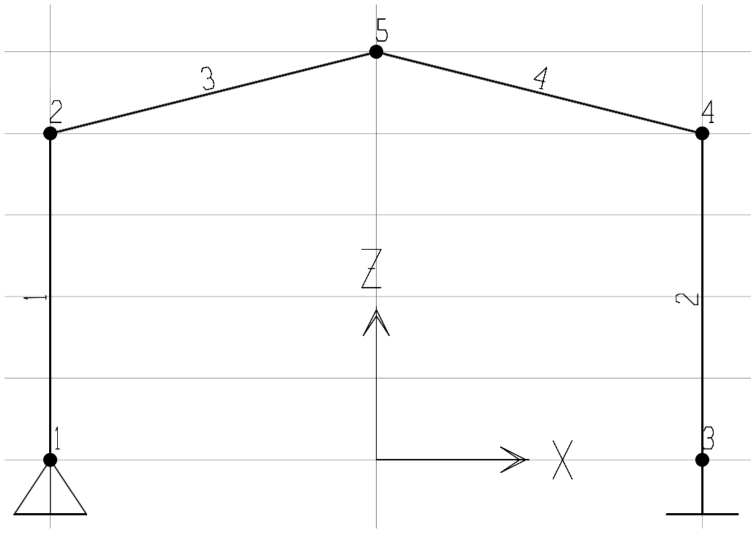
J O I N T   R E A C T I O N S

JOINT	LOAD	F1	F2	F3	M1	M2	M3
1	LOAD1	-10.6271	0.0000	133.6357	0.0000	0.0000	0.0000
3	LOAD1	-69.3729	0.0000	113.7507	0.0000	-149.2315	0.0000

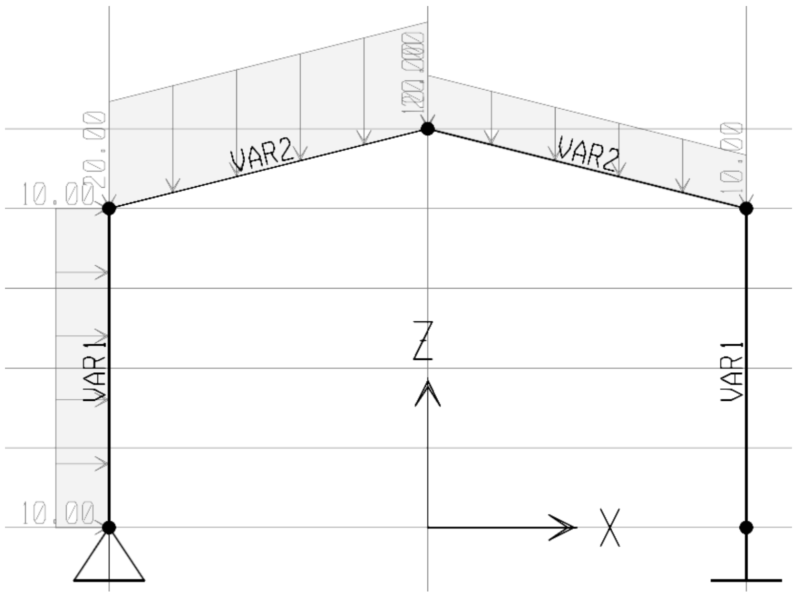
F R A M E   E L E M E N T   F O R C E S

FRAME	LOC	P	V2	V3	T	M2	M3
1	0.00	-133.64	10.63	0.00	0.00	0.00	0.00
	4.00	-133.64	-29.37	0.00	0.00	0.00	37.49
	8.00	-133.64	-69.37	0.00	0.00	0.00	234.98
2	0.00	-113.75	69.37	0.00	0.00	0.00	149.23
	4.00	-113.75	69.37	0.00	0.00	0.00	-128.26
	8.00	-113.75	69.37	0.00	0.00	0.00	-405.75
3	0.00	-59.71	-47.18	0.00	0.00	0.00	35.66
	4.12	-79.71	32.82	0.00	0.00	0.00	65.26
	8.25	-99.71	112.82	0.00	0.00	0.00	-234.98
4	0.00	-74.89	13.53	0.00	0.00	0.00	35.66
	4.12	-84.89	53.53	0.00	0.00	0.00	-102.58
	8.25	-94.89	93.53	0.00	0.00	0.00	-405.75

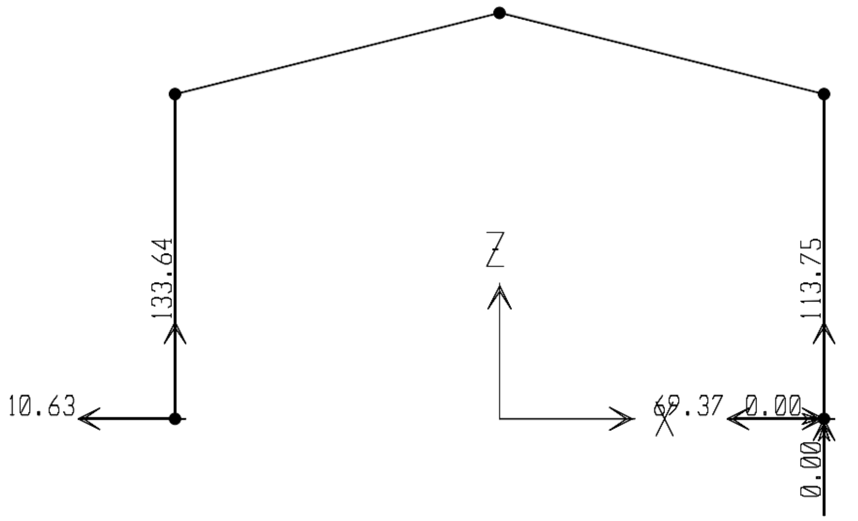
Labels of joints and elements



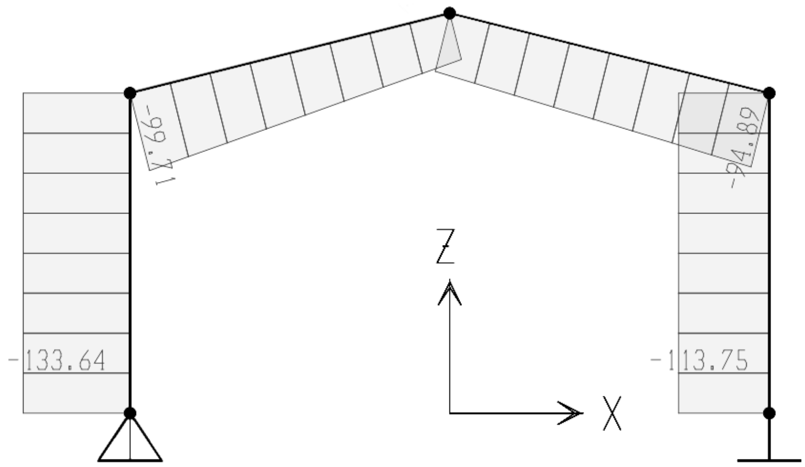
Loads and cross sections



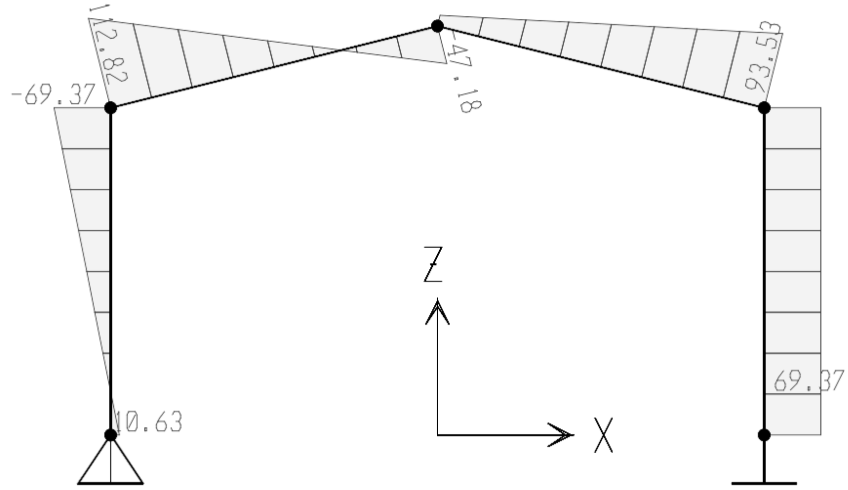
Support reactions



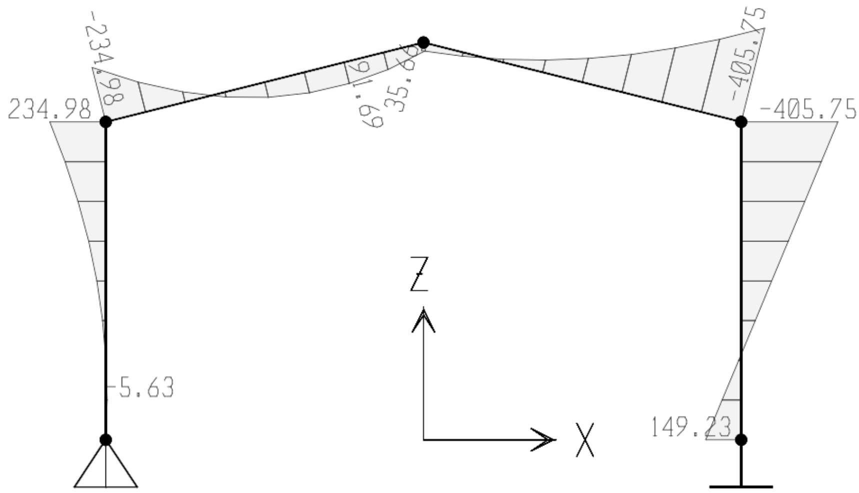
Axial forces



Shear forces



Bending moments



V. Conclusions

The results obtained by the SAP 2000 software match the Calcpad solution with an accuracy of 0.5%. This is likely due to the different way the elastic properties are approximated along the element length.